

Effects of disorder on a lattice Ginzburg-Landau model of d -wave superconductors and superfluids

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We study the effects of quenched disorder on the two-dimensional d -wave superconductors (SCs) at zero temperature by Monte Carlo simulations. The model is defined on the three-dimensional (3D) lattice and the SC pair field is put on each spatial link as motivated in the resonating-valence-bond theory of the high- T_c SCs. For the nonrandom case, the model exhibits a second-order phase transition to a SC state as density of charge carriers is increased. It belongs to the universality class *different from* that of the 3D XY model. Quenched disorders (impurities) are introduced both in the hopping amplitude and the plaquette term of pair fields. Then the second-order transition disappears at a critical concentration of quenched disorder, $p_c \approx 15\%$. Implication of the results to cold atomic systems in optical lattices is also discussed.

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Effects of disorder on phase structures and phase transitions have been studied for various systems. In particular, the high- T_c materials are microscopically highly nonuniform and it is suggested that there exists a spin-glass-like phase near the phase transition point of superconductivity (SC) at low temperatures (T).¹ Furthermore, existence of a Bose glass was recently suggested in the Mott-insulating phase of cold atomic systems in random potentials.² In the present paper, being motivated in part by these observations, we shall study effects of quenched disorders on SCs by using a lattice Ginzburg-Landau (GL) model of unconventional d -wave SCs. The model in its pure case was introduced as the GL theory of the resonating-valence-bond (RVB) field for the t - J model,³ and the case of including interaction with the electromagnetic (EM) field has been investigated recently.⁴ We expect that this model also describes superfluid phase of fermionic atoms in cold atomic systems in optical lattices. There, the effects of disorders can be investigated well under control.⁵

We are interested in quantum SC phase transition of the two-dimensional (2D) model at $T=0$. The model in path-integral representation is described in terms of the following RVB-type Cooper pair field U_{xj} ,

$$U_{xj} \sim \langle \psi_{x+j,\uparrow} \psi_{x\downarrow} - \psi_{x+j,\downarrow} \psi_{x\uparrow} \rangle, \quad (1)$$

where $x(x_0, x_1, x_2)$ is the site of the three-dimensional (3D) cubic lattice of size $V=L^3$ with periodic boundary condition and $j=1, 2$ denotes the spatial direction and also the unit vector in j th direction. $\psi_{x\sigma}$ is the electron annihilation operator at site x and spin $\sigma=\uparrow, \downarrow$. In the $d_{x^2-y^2}$ -wave SC, the Cooper pair amplitudes in $x=x_1$ and $y=x_2$ have the opposite signatures as $\langle U_{x1} U_{x2}^\dagger \rangle < 0$.

Below we shall neglect the effects of the EM field and focus on the effects of quenched disorders. The action of the clean system is then given as follows:

$$\begin{aligned} A = g \sum_x & [c_2 \bar{U}_{x2} U_{x+2,1} \bar{U}_{x+1,2} U_{x1} + c_3 (\bar{U}_{x+1,2} U_{x1} + U_{x+2,1} \bar{U}_{x+1,2} \\ & + \bar{U}_{x2} U_{x+2,1} + U_{x1} \bar{U}_{x2}) + c_4 (\bar{U}_{x+2,1} U_{x1} + \bar{U}_{x+1,2} U_{x2}) \\ & + c_5 (\bar{U}_{x+0,1} U_{x1} + \bar{U}_{x+0,2} U_{x2}) + \text{c.c.}], \quad (2) \end{aligned}$$

where we consider the London limit and set U_{xj} a $U(1)$ variable, $U_{xj} = \exp(i\theta_{xj})$, $\theta_{xj} \in [-\pi, \pi)$. The overall factor g plays a role of $1/\hbar$ and controls *quantum fluctuations*. When we fix c_i , g can be taken as an increasing function of the carrier concentration δ . (In the t - J model δ is the hole density.) The c_2 terms controls fluctuation of flux of U_{xj} 's around each spatial plaquette. The c_3 and c_4 terms represent the spatial hopping of U_{xj} , whereas the c_5 term describes the hopping in the imaginary-time (0th) direction. The partition function Z is given by $Z = \int [dU] \exp(A)$, $[dU] = \prod_{x,j} d\theta_{xj} / (2\pi)$. We consider the parameter region $c_3 < 0$ to expect $\langle U_{x1} U_{x2}^\dagger \rangle < 0$, although $Z(c_3) = Z(-c_3)$ because of the change of variables $U_{x1} \rightarrow -U_{x1}$. The action A is related to the action A_{Higgs} of the $U(1)$ Higgs gauge theory that is obtained from Eq. (2) by the replacement $U_{xj} \rightarrow \bar{\phi}_{x+j} U_{xj} \phi_x$, where $\phi_x = \exp(i\varphi_x)$ is the $U(1)$ Higgs field. A_{Higgs} is invariant under time-independent local gauge transformation $\varphi_x \rightarrow \varphi_x + \lambda_x$, $\theta_{xj} \rightarrow \theta_{xj} + \lambda_{x+j} - \lambda_x$, where $\lambda_{x(x_1, x_2)}$ is an x_0 -independent function. Actually, A is viewed as the gauge-fixed version of A_{Higgs} in the unitary gauge $\phi_x = 1$.

Quenched disorder is introduced in the system (2) by replacing the coefficients $c_{2,3,4}$ as spatial-plaquette dependent ones. First, we consider the 2D spatial plane with fixed x_0 , say $x_0=0$. Among L^2 plaquettes in the plane we choose $p \times L^2$ plaquettes randomly as ones at which impurities reside. We call it a sample. We consider that the configuration of "wrong" plaquettes is x_0 independent because the location of impurities is fixed along the imaginary time. Then we reverse the values of $c_{2,3,4}$ for interaction terms contained in these plaquettes.⁶ Thus the new plaquette-dependent coefficients $c_{2,3,4}^p$ are given by

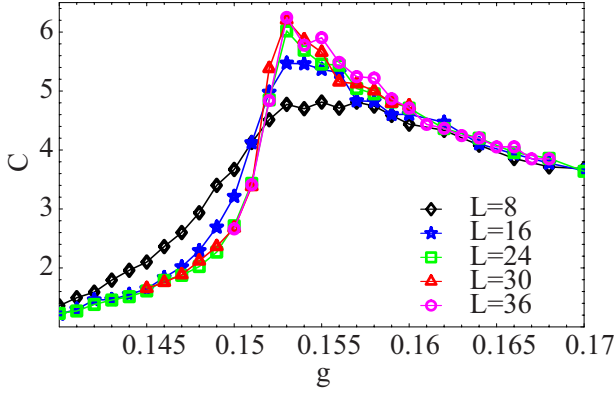


FIG. 1. (Color online) Specific heat C for $p=0$ as a function of g with $c_2=-c_3=c_4=c_5=1$. Error bars represent MC errors. System-size dependence of its peak supports existence of a second-order phase transition.

$$c_{2,3,4}^p = \begin{cases} c_{2,3,4} & \text{with probability } 1-p \\ -c_{2,3,4} & \text{with probability } p \end{cases} \quad (3)$$

for each spatial plaquette. Please note $c_5^p=c_5$. Then the partition function Z_p and the free energy F_p per site of one sample are given by

$$Z_p = \int [dU] \exp(A_p) = \exp(-VF_p), \quad (4)$$

where A_p is obtained from Eq. (2) by replacing c_i by c_i^p . To obtain an ensemble average $\langle O \rangle$ of an observable $O(\{U_{xy}\})$ in the disordered system, we first prepare N_s samples and calculate the quantum-mechanical average $\langle O \rangle_s$ for the s th sample ($s=1, \dots, N_s$). Then we average it over samples

$$\langle O \rangle = \frac{1}{N_s} \sum_{s=1}^{N_s} \langle O \rangle_s = Z_p^{-1} \int [dU] O \exp(A_p). \quad (5)$$

For the MC simulations, we used the standard METROPOLIS algorithm.⁷ The typical statistics used was 10^5 sweeps per block and the average and MC errors were estimated over ten blocks for each sample. Then we take quenched averages over $N_s=30 \sim 50$ samples. We estimated standard deviation

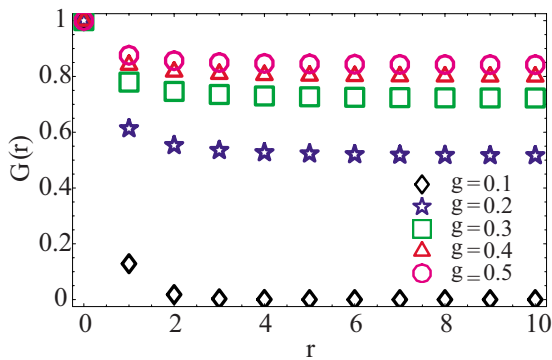


FIG. 2. (Color online) Correlation function $G(r)$ for various values of g with $c_{2,4,5}=-c_3=1$. Critical coupling g_c is estimated $g_c=0.154$ for $L=20$. Data show that for $g > g_c$ there exists SC LRO, whereas $g < g_c$ no LRO.

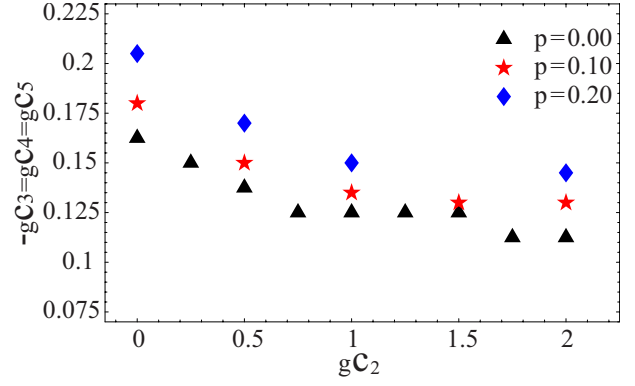


FIG. 3. (Color online) Location of the peak of C with $L=12$ and $N_s=30$ in the $gc_2-g \times (-c_3=c_4=c_5)$ plane for $p=0, 0.10, 0.20$. The region of disorder “phase” increases as p increases.

of physical quantities such as “specific heat” over samples (we call it sample error) as a function of N_s , which becomes stable for $N_s \gtrsim 30$.

Let us first consider the nonrandom case, $p=0$. We studied the phase structure by calculating the “internal energy” per site $E=-\langle A \rangle/V$ and the “specific heat” per site $C=\langle (A-\langle A \rangle)^2 \rangle/V$. Typical behavior of C is shown in Fig. 1, which indicates a second-order phase transition at $g=g_c \approx 0.154$. This transition has been predicted in Ref. 3 and also observed in the presence of the EM gauge interactions.⁴ In order to verify that it is a transition to a SC phase, we measured the correlation function of U_{xj} , the order parameter field of SC

$$G(r) = \frac{1}{4V} \sum_{x,i \neq j} \langle \bar{U}_{xj} U_{x+ri,j} \rangle + c.c. \quad (6)$$

We present $G(r)$ in Fig. 2. It is obvious that there exists SC long-range order (LRO) for $g > g_c$, whereas no LRO for $g < g_c$.⁸ Therefore the observed transition is nothing but the SC phase transition.

The critical exponents for $c_{2,4,5}=-c_3=1$ were estimated by the finite-size scaling analysis for C . We obtained $\nu=1.5$, $\alpha=0.285$, and the critical coupling $g_\infty=0.153$. When $c_2=c_3=0$, the system (2) reduces to a set of decoupled 2D XY spin models [U_{xj} plays the role of an XY spin], each of which has the Kosterlitz-Thouless transition. The c_2 and c_3 terms couple these 2D XY spins. From the above values of critical exponents, we judge that the present phase transition does not belong to the universality class of the 3D XY model having $\nu=0.6721(13)$.⁹

Let us next turn to the random case, $p > 0$. At first, we consider the gc_2-gc' plane, where $c' \equiv -c_3=c_4=c_5$, and search for the location of the peak of C . In Fig. 3, we present the peak location of C for given disorder concentration $p=0.0, 0.10, 0.20$. For $p=0.0$, the peak line expresses the second-order transition as we saw in Figs. 1 and 2. We see that the region of the normal (non-SC) state increases as

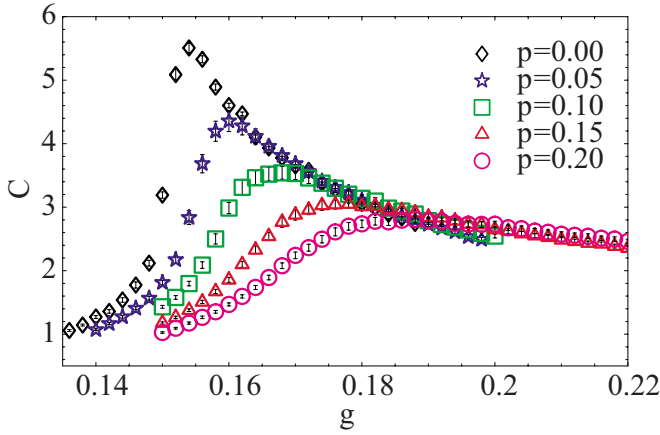


FIG. 4. (Color online) Specific heat C for $L=16$ with $N_s=30$ vs g for $p=0.0, \dots, 0.20$. Error bars here and below represent sample errors. The peak around the SC phase transition is getting weaker as p is increased. Location of the peak moves to larger g .

p increases, although we need to check whether the location of this peak expresses genuine SC transition for $p > 0$. Below we examine it by focusing on the specific case $c_2 = -c_3 = c_4 = c_5 = 1$ with the varying parameter g and calculate E , C , $G(r)$, etc. Density of quenched disorder that we studied is $p=0.05, 0.10, 0.15$, and 0.20 .

We first present the result of C in Fig. 4. The signal of the second-order phase transition at $p=0$ is getting weaker as p is increased, and also the location of the peak moves to larger g . Next, in Fig. 5, we present the system-size dependence (SSD) of C for $p=0.10, 0.15$, and 0.20 . We also calculated the derivative of C , $D(g) \equiv gdC(g)/dg$, to identify the order of the phase transition. Results are shown in Fig. 6. From these results, we judge that there exists a second-order transition for $p \leq 0.10$ whereas it changes to a crossover for $p \geq 0.20$. The case of $p=0.15$ seems to have no SSD in C but have certain SSD for $D(g)$. Therefore, the transition in the $p=0.15$ case might be of third order.

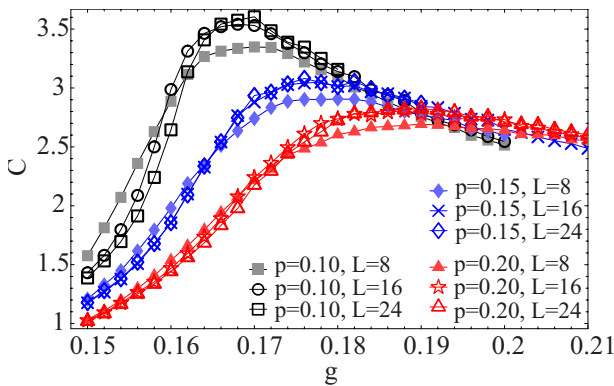


FIG. 5. (Color online) The system size dependence of C with $N_s=30$ for $p=0.10, 0.15, 0.20$. C for $p=0.10$ shows SSD, which supports a second-order phase transition to SC state. C for $p=0.15$ has less SSD, and C for $p=0.20$ shows no SSD for $L=16, 24$.

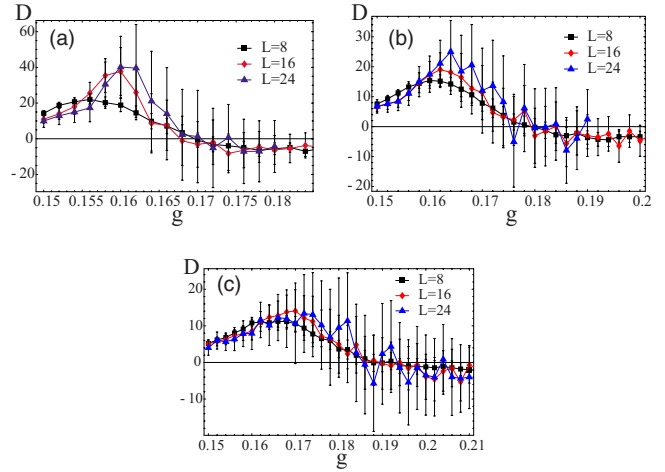


FIG. 6. (Color online) $D(g) \equiv gdC(g)/dg$ with $N_s=30$ vs g for various L . (a) $p=0.10$; (b) $p=0.15$; (c) $p=0.20$. Cases (a) and (b) show SSD, while the case (c) shows almost no SSD.

To study if the SC state persists even for $p \geq 0.20$, we measured the SC correlation $G(r)$ averaged over samples with randomly generated c_i^p . In Fig. 7 we present the results for $p=0.10, 0.15$, and 0.20 . It supports that for $p=0.10$ there exists the SC LRO for $g > g_c$ whereas for $p=0.20$ no LRO for any values of g . This observation leads to the conclusion that the SC phase disappears for $p \geq 0.20$, i.e., there is a multicritical point p_{mc} near $p=0.15$ in the $g-p$ phase diagram.

It is interesting to ask what kind of phase is realized in the region $p > p_{mc}$ and $g > g_{cr}$, where g_{cr} is the crossover coupling determined by the peak location of the specific heat. Probably, in the case of the doped AF magnets with strong inhomogeneity, this phase may simply correspond to dirty

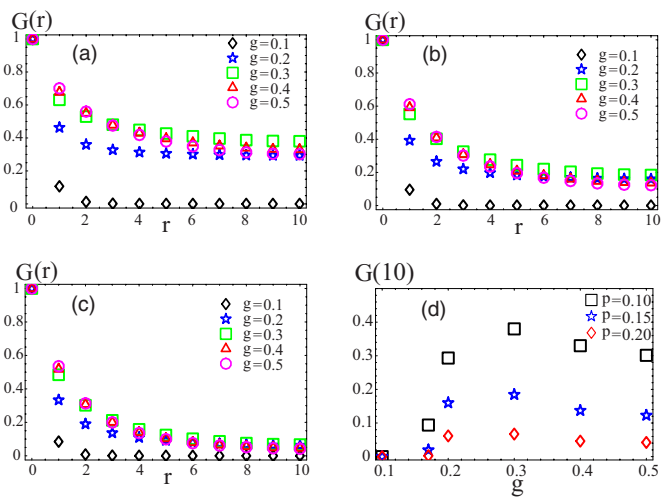


FIG. 7. (Color online) SC correlation function $G(r)$ for $L=20$ with $N_s=50$. (a) $p=0.10$; (b) $p=0.15$; (c) $p=0.20$. (d) $G(r=10)$ for (a)–(c) vs g . SC “transition point” g_c is estimated from the specific heat C as $g_c \simeq$ (a) 0.168 , (b) 0.17 , (c) 0.18 . (d) supports that LRO develops for $g > g_c$ in (a).

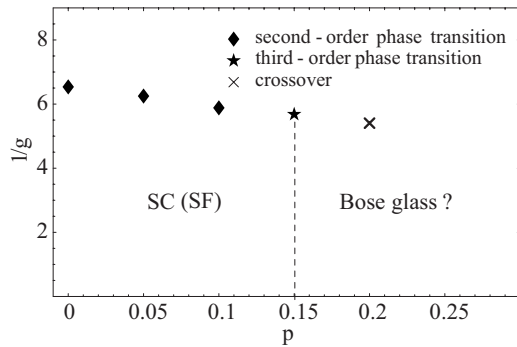


FIG. 8. Phase diagram in the $p-g^{-1}$ phase diagram for the lattice model of dirty d -wave SC with $c_2=-c_3=c_4=c_5=1$. Possible Bose glass phase is suggested. There should be a multicritical point near the line $p=0.15$.

“normal” metallic state. We think that the present model given by Eq. (2) also describes the superfluid d -wave RVB

state of fermionic atoms in 2D optical lattice, which was recently proposed by several authors.¹⁰ Random disorders can be introduced into the systems by a laser speckle or by an incommensurate bichromatic potential. In that experimental setup, there is an interesting possibility that a Bose glass phase, which is an analog of the spin glass phase, is realized there. For ultracold strongly interacting ^{87}Rb bosons, a Bose glass phase is suggested by experiments.² In the experiments of cold atom systems in optical lattices, the Bose glass phase has no long-range coherence but excitations are gapless. In Fig. 8, an expected phase diagram of the present lattice model of dirty d -wave SC and superfluidity (SF) is shown. We think that existence of the Bose glass phase is examined theoretically by the standard analytical and numerical methods applied for spin glass.^{11,12}

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⁶One wrong plaquette gives rise to four c_3 terms, two c_4 terms, and one c_2 term (except for their $c.c.$) with the sign-reversed coefficients.

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